A Variation-of-Constants Formula for Abstract Functional Differential Equations in the Phase Space

Yoshiyuki Hino
Department of Mathematics and Informatics, Chiba University,
1-33 Yayoi-cho, Inageku, Chiba 263-8522, Japan
E-mail: hino@math.s.chiba-u.ac.jp

Satoru Murakami
Department of Applied Mathematics, Okayama University of Science,
1-1 Ridaicho, Okayama 700-0005, Japan
E-mail: murakami@youhei.xmath.ous.ac.jp

Toshiki Naito
Department of Mathematics, The University of Electro-Communications,
Chofu, Tokyo 182-8585, Japan
E-mail: naito@e-one.uec.ac.jp

and

Nguyen Van Minh
Department of Mathematics, Hanoi University of Science,
Khoa Toan, Dai Hoc Khoa Hoa Tu Nhien, 334 Nguyen Trai, Hanoi, Vietnam
E-mail: nvminh@netnam.vn

Received July 20, 2000; revised November 22, 2000

For linear functional differential equations with infinite delay in a Banach space, a variation-of-constants formula is established in the phase space. As an application one applies it to study the admissibility of some spaces of functions whose spectra are contained in a closed subset of the real line.

Key Words: functional differential equations; phase space; variation-of-constants formula; spectrum of functions; admissibility.

1. INTRODUCTION

In this paper we are concerned with the linear functional differential equation with infinite delay

\[ \dot{u}(t) = Au(t) + L(u_t) + f(t), \quad u(t) \in X, \]  

(1)

To whom all correspondence should be addressed.
on a phase space $B = \mathcal{B}((−∞, 0]; X)$ satisfying some fundamental axioms stated in Section 2, where $X$ is a Banach space, $A$ is the infinitesimal generator of a strongly continuous semigroup on $X$, $u_t$ is the element of $\mathcal{B}((−∞, 0]; X)$ defined by $u_t(s) = u(t+s)$ for $s \in (−∞, 0]$, and $L$ is a bounded linear operator mapping $B$ into $X$.

The main purpose of this paper is to establish a representation formula for solutions of (1) in the phase space $B$ which corresponds to the variation-of-constants formula in the theory of linear ordinary differential equations. So, we often call the representation formula obtained here the variation-of-constants formula in the phase space. Such a representation formula is a powerful tool which is widely used in various studies for the qualitative theory of differential equations and functional differential equations; see e.g. [2, 3, 5, 6, 10, 14–16, 24, 25, 28] and the references therein. When $X$ is finite dimensional, the representation formula for functional differential equations has been established by Hale [5] in the case of finite delay and by Murakami [16] in the case of infinite delay. In the infinite dimensional case, however, there arise some difficulties in establishing the representation formula for (1) in the phase space. In fact, in the finite dimensional case, the adjoint equation of the homogeneous equation associated with (1) has essentially been utilized (cf. [16]). Up to now, in the infinite dimensional case, however, the adjoint theory has not been developed well enough to establish the formula for (1). Of course, the representation formula in $X$ can be easily established even in the infinite dimensional case. However, it is not the case for the formula in the phase space. Actually, in the infinite dimensional case, the representation formula in the phase space for functional differential equations with finite delay has been treated in several works (see e.g. [2, 3, 6, 15, 24, 25, 28] and the references therein). However, it seems that the formula obtained in [15, 28] is not exactly the one in the phase space as claimed\(^2\). We notice that in [1] a variation of constants formula has been discussed for the bounded case, i.e., the case in which the operator $A = 0$. It may be seen that the method employed in [1] is obviously based on the boundedness of the equation, and hence it is unapplicable to the unbounded case, i.e., the case $A \neq 0$. Other alternative approaches to the problem can be found in [2, 3, 6, 24, 25]. Especially, in [2, 3] the perturbation theory of semigroups has been extensively developed and a variation-of-constants formula has been established in an extended space, involving sun-star spaces.

\(^2\) In general, the solution semigroup is not defined at discontinuous functions. If one extends its domain to this function space as done in [15] or [28, p. 115], then this semigroup is not strongly continuous even in the simplest case. So, in this way the integral in the formula is undefined as an integral in the phase space.
In this paper, we shall make an attempt to clarify the ambiguity in the variation of constants formula discussed in [15, 28] by establishing a representation formula in the phase space and the decomposition of the formula to the stable subspace or the unstable subspace. One of the crucial points in our approach to the formula is not to treat the adjoint equation, but to approximate solutions in terms of some “nice” elements of the phase space by using the principle of superposition for (1). Therefore, our approach developed in this paper is quite simpler than the one in [2, 3, 5, 14, 16].

As an application of our formula, we shall investigate the admissibility of some spaces of functions whose spectra are contained in a closed subset of $\mathbb{R}$. The main conditions found are stated in terms of the spectral properties of the characteristic operator associated with the linear homogeneous equation. These conditions are sharper than those in [4, 17] in the case where $A$ generates a compact semigroup and $B$ is uniformly fading memory. Further applications of the formula will be the subject of our future investigation.

2. PHASE SPACE $B$

In this section we shall define the phase space $B$ which is employed throughout the paper.

Let $X$ be a complex Banach space with norm $|\cdot|$. For any interval $J \subset \mathbb{R} := (-\infty, \infty)$, we denote by $C(J; X)$ the space of all continuous functions mapping $J$ into $X$. Moreover, we denote by $BC(J; X)$ the subspace of $C(J; X)$ which consists of all bounded functions. Clearly, $BC(J; X)$ is a Banach space with the norm $|\cdot|_{BC(J; X)}$ defined by $|f|_{BC(J; X)} = \sup\{|f(t)| : t \in J\}$. If $J = \mathbb{R}$, then we simply write the norm $|\cdot|_{BC(J; X)}$ of the Banach space $BC(J; X)$ as $||\cdot||$. For any function $x: (-\infty, a) \mapsto X$ and $t < a$, we define a function $x_t: \mathbb{R}^- := (-\infty, 0] \mapsto X$ by $x_t(s) = x(t+s)$ for $s \in \mathbb{R}^-$. Let $B = B(\mathbb{R}^-; X)$ be a complex linear space of functions mapping $\mathbb{R}^-$ into $X$ with a complete seminorm $|\cdot|_a$. The space $B$ is assumed to have the following properties:

(A1) There exist a positive constant $N$ and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on $R^+ := [0, \infty)$ with the property that if $x: (-\infty, a) \mapsto X$ is continuous on $[\sigma, a)$ with $x_\sigma \in B$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,

\begin{itemize}
  \item[(i)] $x_t \in B$,
  \item[(ii)] $x_t$ is continuous in $t$ (w.r.t. $|\cdot|_a$),
  \item[(iii)] $N|x(t)| \leq |x_\sigma|_a \leq K(t-\sigma) \sup_{s \in [\sigma, t]} |x(s)| + M(t-\sigma) |x_\sigma|_a$.
\end{itemize}

(A2) If $\{\phi^k\}, \phi^k \in B$, converges to $\phi$ uniformly on any compact set in $\mathbb{R}^-$ and if $\{\phi^k\}$ is a Cauchy sequence in $B$, then $\phi \in B$ and $\phi^k \to \phi$ in $B$. 

The space $\mathscr{B}$ is called a uniform fading memory space, if it satisfies (A1) and (A2) with $K(\cdot) \equiv K$ (a constant) and $M(\beta) \to 0$ as $\beta \to \infty$ in (A1). A typical example of uniform fading memory spaces is

$$C_r := C_r(X) = \{ \phi \in C(\mathbb{R}^+; X) : \lim_{\theta \to -\infty} \phi(\theta) e^{\gamma \theta} = 0 \}$$

which is equipped with norm $|\phi|_{C_r} = \sup_{\theta < 0} |\phi(\theta)| e^{\gamma \theta}$, where $\gamma$ is a positive constant.

It is known [7, Lemma 3.2] that if $\mathscr{B}$ is a uniform fading memory space, then $BC := BC(\mathbb{R}^+; X) \subset \mathscr{B}$ and the inclusion map from $BC$ into $\mathscr{B}$ is continuous. For other properties of uniform fading memory spaces, we refer the reader to the book [10].

3. VARIATION-OF-CONSTANTS FORMULA IN THE PHASE SPACE

In this section we first assume that the space $\mathscr{B} = \mathscr{B}(\mathbb{R}^+; X)$ satisfies (A1). We then consider the (nonhomogeneous) linear functional differential equation

$$\dot{u}(t) = Au(t) + L(u_t) + f(t),$$

(2)

where $f \in BC(\mathbb{R}; X)$, $A$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$, and $L$ is a bounded linear operator mapping the space $\mathscr{B} = \mathscr{B}(\mathbb{R}^+; X)$ into $X$. Throughout the paper we shall assume that the operator $L$ is of the form

$$L(\phi) = \int_0^\theta \frac{d}{d\theta} \eta(\theta) \phi(\theta), \quad \phi \in C_{00},$$

where $\eta(\theta)$ is a $B(X)$-valued function of locally bounded variation on $\mathbb{R}^-$ which is left continuous in $\theta < 0$ with $\eta(0) = 0$; here $C_{00}$ denotes the subspace of $C(\mathbb{R}^+; X)$ consisting of functions with compact support, and $B(X)$ is the space of all bounded linear operators on $X$. For any $(\sigma, \phi) \in \mathbb{R} \times \mathscr{B}$, there exists a (unique) function $u: \mathbb{R} \to X$ such that $u_\sigma = \phi$, $u$ is continuous on $[\sigma, \infty)$, and the following relation holds:

$$u(t) = T(t - \sigma) \phi(0) + \int_\sigma^t T(t - s) \{ L(u_s) + f(s) \} \, ds, \quad t \geq \sigma,$$

(cf. [9, Theorem 1]). The function $u$ is called a (mild) solution of (2) through $(\sigma, \phi)$ on $[\sigma, \infty)$ and denoted by $u(\cdot, \sigma, \phi; f)$. Also, a function $v \in C(\mathbb{R}; X)$
is called a (mild) solution of (2) on $\mathbb{R}$ if $v_t \in \mathcal{B}$ for all $t \in \mathbb{R}$ and it satisfies

$$u(t, \sigma, v_\sigma; f) = v(t) \quad \text{all} \quad t \quad \text{and} \quad \sigma \quad \text{with} \quad t \geq \sigma.$$  

For any $t \geq 0$, we define an operator $V(t)$ on $\mathcal{B}$ by

$$V(t) \phi = u_t(0, \phi; 0), \quad \phi \in \mathcal{B}.$$  

We can easily see that $(V(t))_{t \geq 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{B}$, which is called the solution semigroup of (2). By the principle of superposition, we get the relation

$$u_t(\sigma, \phi; f) = u_t(\sigma, \phi; 0) + u_t(\sigma, 0; f)$$

$$= V(t-\sigma) \phi + u_t(\sigma, 0; f). \quad (3)$$

In what follows, we shall give a representation of $u_t(\sigma, 0; f)$ in terms of $f$ and the solution semigroup $(V(t))_{t \geq 0}$. To this end, we introduce a function $I^n$ defined by

$$I^n(\theta) = \begin{cases} (n\theta + 1) \cdot I, & -1/n \leq \theta \leq 0 \\ 0, & \theta < -1/n, \end{cases}$$

where $n$ is any positive integer and $I$ is the identity operator on $X$. It follows from (A1) that if $x \in X$, then $I^n x \in \mathcal{B}$ with $|I^n x|_{\mathcal{B}} \leq K(1) |x|$. Moreover, the $\mathcal{B}$-valued function $V(t-s) I^n f(s)$ is continuous in $s \in (-\infty, t]$ whenever $f \in BC(\mathbb{R}, X)$.

The following theorem yields a representation formula for solutions of (2) in the phase space:

**Theorem 3.1.** The segment $u_t(\sigma, \phi; f)$ of solution $u(\cdot, \sigma, \phi, f)$ of (2) satisfies the following relation in $\mathcal{B}$:

$$u_t(\sigma, \phi; f) = V(t-\sigma) \phi + \lim_{n \to \infty} \int_0^t V(t-s) I^n f(s) \, ds.$$  

For the proof of the theorem, we need some lemmas:

**Lemma 3.1.** There exists a unique $\mathcal{B}(X)$-valued function $W(t)$, $W(0) = I$, on $\mathbb{R}^+$ such that for any $x \in X$, $v(t) := W(t) x$ is continuous in $t \geq 0$ and

$$v(t) = T(t) x + \int_0^t T(t-s) \left( \int_{-s}^{\theta} d\eta(\theta) v(s+\theta) \right) ds. \quad (4)$$

**Proof.** Consider the function $y$ defined by $y_0 \equiv 0$ and $y(t) = v(t) - x$ for $t \geq 0$. Then Eq. (4) for $v(t)$ is reduced to

$$y(t) + x = T(t) x + \int_0^t T(t-s) \left( \int_{-s}^{\theta} d\eta(\theta) y(s+\theta) + \int_{-s}^{\theta} d\eta(\theta) x \right) ds.$$
or

\[ y(t) = (T(t) - I)x - \int_{0}^{t} T(t-s) \eta(-s) x \, ds + \int_{0}^{t} T(t-s) L(y_s) \, ds. \]

The above equation for \( y \) possesses a unique solution. Indeed, this can be proved by Picard’s iteration method, so the details are omitted.

For any \( x \in X \), we put

\[ v(t; x) = W(t) x \quad (t \geq 0), \quad v(t, x) = 0 \quad (t < 0). \]

Clearly, the function \( v(t-s; h(s)) \) is continuous in \((t, s), t \geq s\), whenever \( h \in BC(R; X) \).

**Lemma 3.2.** Let \( h \in BC(R; X) \). Then \( \int_{\sigma}^{t} v(t-s; h(s)) \, ds = u(t, \sigma, 0; h) \).

**Proof.** The above relation can be established by almost the same calculation as in the proof of [28, Theorem 4.2.1]. Indeed, if we set \( z(t) = \int_{\sigma}^{t} v(t-s; h(s)) \, ds \), then

\[
\int_{\sigma}^{t} T(t-s) [L(z_s) + h(s)] \, ds
\]

\[
= \int_{\sigma}^{t} T(t-s) h(s) \, ds + \int_{\sigma}^{t} T(t-s) \left( \int_{-\infty}^{0} d\eta(\theta) z(s+\theta) \right) ds
\]

\[
= \int_{\sigma}^{t} T(t-s) h(s) \, ds
\]

\[
+ \int_{\sigma}^{t} T(t-s) \left( \int_{\sigma-\chi}^{\sigma+\theta} d\eta(\theta) \int_{\sigma-\chi}^{t-\theta} v(s+\theta-\chi; h(\chi)) \, d\chi \right) ds
\]

\[
= \int_{\sigma}^{t} T(t-s) h(s) \, ds
\]

\[
+ \int_{\sigma}^{t} T(t-s) \left( \int_{\sigma-\chi}^{\sigma+\theta} d\eta(\theta) v(s+\theta-\chi; h(\chi)) \, d\chi \right) ds
\]

\[
= \int_{\sigma}^{t} T(t-s) h(s) \, ds
\]

\[
+ \int_{\sigma}^{t} \left( \int_{\sigma-\chi}^{\sigma+\theta} d\eta(\theta) v(s+\theta-\chi; h(\chi)) \right) ds
\]
\[
\begin{align*}
&= \int_{\sigma}^{t} T(t-s) h(s) \, ds \\
&+ \int_0^{\sigma} \left( \int_{-\infty}^{t-s} T(t-\chi-w) \int_{-w}^{0} \, \mathrm{d}n(\theta) \, v(w+\theta; h(\chi)) \, dw \right) \, d\chi \\
&= \int_{\sigma}^{t} T(t-s) h(s) \, ds + \int_{\sigma}^{t} (v(t-\chi; h(\chi)) - T(t-\chi) h(\chi)) \, d\chi \\
&= \int_{\sigma}^{t} v(t-\chi; h(\chi)) \, d\chi \\
&= z(t)
\end{align*}
\]

for \( t > \sigma \). Also, if \( t \leq \sigma \), then \( z(t) = 0 = u(t, \sigma; 0, h) \). This completes the proof.

**Lemma 3.3.** \( u(t, 0, \Gamma^n x; 0) \rightarrow v(t; x) \) as \( n \rightarrow \infty \), uniformly for each bounded \( (t, x) \in \mathbb{R}^+ \times X \).

**Proof.** Let \( u^*(t) = u(t, 0, \Gamma^n x; 0) \) for \( t \geq 0 \). Then

\[
|u^*(t) - v(t; x)| = \left| \int_0^{t} \left( \int_{-s-1/n}^{-s} \, d\eta(\theta) \, u^*(s+\theta) \right) \, ds \right|
\]

\[
= \left| \int_0^{t} \left( \int_{-\infty}^{-1/n} \, d\eta(\theta) \, u^*(s+\theta) \right) \, ds \right|
\]

\[
\leq C_t \int_0^{t} \left( \text{Var}(\eta; [\sigma, 0]) \, f^n(s, x) + e(n, s) |x| \right) \, ds,
\]

where

\[
f^n(t, x) = \sup_{0 \leq \tau \leq t} |u^*(\tau) - v(\tau; x)|,
\]

\[
C_t = \sup_{0 \leq \tau \leq t} \|T(\tau)\|,
\]

\[
e(n, s) = \text{Var}(\eta; [-s-1/n, -s])
\]
and \( \text{Var}(\eta; J) \) denotes the total variation of \( \eta \) on an interval \( J \). Hence

\[
f^*(t, x) \leq C_t \text{Var}(\eta; [-t, 0]) \int_0^t f^*(s, x) \, ds + C_t |x| \int_0^t \varepsilon(n, s) \, ds,
\]

and consequently

\[
f^*(t, x) \leq C^* |x| \int_0^t \varepsilon(n, s) \, ds
\]

by Gronwall’s inequality, where \( C^* \) is a constant depending only on \( t \). We claim that

\[
\lim_{\delta \to 0} \text{Var}(\eta; [-s-\delta, -s]) = 0
\]

for \( s > 0 \). If the claim holds true, then \( \varepsilon(n, s) \to 0 \) for \( s > 0 \) as \( n \to \infty \). Then Lebesgue’s convergence theorem yields that \( f^*(t, x) \to 0 \) as \( n \to \infty \), uniformly for each bounded \((t, x)\). Now, in what follows we shall establish the above claim. Assume the claim is not true. Then for some \( s > 0 \) there is a constant \( c > 0 \) such that \( \text{Var}(\eta; [-s-\delta, -s]) > c \) for all \( \delta > 0 \). In particular, since \( \text{Var}(\eta; [-s-1, -s]) > c \), there is a partition \( t_0 = -s-1 < t_1 < \cdots < t_N = -s \) such that \( \sum_{k=0}^{N-1} \|\eta(t_k) - \eta(t_{k-1})]\| > c \). Since \( \eta \) is left continuous at \(-s\), we may assume that \( \|\eta(t_{k-1}) - \eta(-s)\| < c/2 \) by taking \( t_{k-1} \) close to \(-s\) if necessary. Then

\[
\text{Var}(\eta; [-s-1, a_1]) \geq \sum_{k=0}^{N-1} \|\eta(t_k) - \eta(t_{k-1})\| \geq c/2,
\]

where \( a_k := t_{k-1} \). Notice that \( \text{Var}(\eta; [a_1, -s]) > c \) by the assumption. Employing the same reasoning as above, one can see that \( \text{Var}(\eta; [a_1, a_2]) \geq c/2 \) for some \( a_2 \in (a_1, -s) \). Repeat this procedure to get a sequence \( \{a_k\} \) such that \( a_0 := -s-1 < a_1 < a_2 < \cdots < -s \) and \( \text{Var}(\eta; [a_k, a_{k+1}]) \geq c/2 \) for \( k = 0, 1, 2, \ldots \). Then

\[
\text{Var}(\eta; [-s-1, -s]) \geq \text{Var}(\eta; [a_0, a_m]) = \sum_{k=0}^{m-1} \text{Var}(\eta; [a_k, a_{k+1}]) \geq cm/2 \to \infty
\]

as \( m \to \infty \). This is a contradiction to \( \text{Var}(\eta; [-s-1, -s]) < \infty \). Thus, the claim is proved. This completes the proof of the lemma.
Finally we shall prove the following lemma; from (3) and this lemma Theorem 3.1 follows immediately.

**Lemma 3.4.** \( \lim_{n \to \infty} \int_{\sigma}^{t} V(t-s) \Gamma^n f(s) \, ds = u_t(\sigma, 0; f) \) in \( B \).

**Proof.** The integral \( \int_{\sigma}^{t} V(t-s) \Gamma^n f(s) \, ds \) is the limit of a Riemann sum of the form \( \phi^d := \sum_k V(t-s_k) \Gamma^n f(s_k) \, ds_k \) in \( B \). Observe that \( \phi^d(\theta) = \sum_k u(t-s_k+\theta, 0, \Gamma^n f(s_k); 0) \, ds_k \) is a Riemann sum of the integral

\[
\int_{\sigma}^{t} u(t-s+\theta, 0, \Gamma^n f(s); 0) \, ds =: \xi^n(\theta)
\]

and it converges to the integral uniformly on any compact set in \( R \). Since \( \xi^n(\theta) \) is continuous in \( \theta \leq 0 \) with \( \xi^n(\theta) = 0 \) for \( \theta \leq \sigma - t - 1/n \), it follows from (A1)(i) that \( \xi^n \in B \). Moreover, we get

\[
|\xi^n - \phi^d|_B \leq K_1 \sup_{\sigma - 1/n \leq \theta \leq 0} |\xi^n(\theta) - \phi^d(\theta)|
\]

by (A1)(iii), where \( K_1 = K(t-\sigma+1) \). Thus \( \phi^d \) converges to \( \xi^n \) in \( B \), and hence

\[
\left| \int_{\sigma}^{t} V(t-s) \Gamma^n f(s) \, ds - \xi^n \right|_B = 0.
\]

Using (A1)(iii) again, we get

\[
\left| u_t(\sigma, 0; f) - \int_{\sigma}^{t} V(t-s) \Gamma^n f(s) \, ds \right|_B
\]

\[
= |u_t(\sigma, 0; f) - \xi^n|_B
\]

\[
\leq K_1 \sup_{\sigma - 1/n \leq \theta \leq 0} |u(t+\theta, \sigma, 0; f) - \xi^n(\theta)|.
\]

On the other hand, Lemma 3.3 implies that

\[
\lim_{n \to \infty} \xi^n(\theta) = \int_{\sigma}^{t} v(t-s+\theta; f(s)) \, ds
\]

\[
= \int_{\sigma}^{t+\theta} v(t-s+\theta; f(s)) \, ds
\]

\[
= u(t+\theta, \sigma, 0; f)
\]

uniformly for \( \theta \in [\sigma - t - 1/n, 0] \). Hence, the lemma is proved. \( \square \)
4. DECOMPOSITION OF VARIATION-OF-CONSTANTS FORMULA

Let us consider the case where the space $\mathcal{B}$ is decomposed as the direct sum of two closed subspaces $E_1$ and $E_2$ which are invariant under the solution semigroup $(V(t))_{t \geq 0}$:

$$\mathcal{B} = E_1 \oplus E_2, \quad V(t)E_i \subset E_i \quad (i = 1, 2; t \geq 0).$$

Denote by $\Pi^E_i$ the projection on $E_i$ which corresponds to the above decomposition. It follows from the invariance of $E_i$ under $V(t)$ that

$$\Pi^E_i V(t) = V(t) \Pi^E_i \quad (i = 1, 2).$$

Since the projection $\Pi^E_i$ is continuous on $\mathcal{B}$, we get the following decomposition of the variation-of-constants formula; here and hereafter we denote by $V^E_i(t)$ the restriction of the operator $V(t)$ to $E_i$ and $\phi^E_i = \Pi^E_i \phi$ for $\phi \in \mathcal{B}$:

**Theorem 4.1.** Assume that $\mathcal{B}$ is decomposed as cited above. Then the segment $u(t, \sigma; \phi; f)$ of solution $u(\cdot, \sigma, \phi, f)$ of (2) satisfies the following relation in $\mathcal{B}$:

$$\Pi^E_i u(t, \sigma; \phi; f) = V^E_i(t-\sigma) \phi^E_i + \lim_{n \to \infty} \int_\sigma^t V^E_i(t-s) \Pi^E_i (I^n f(s)) \, ds$$

for $i = 1, 2$.

Let $\xi_i: \mathbb{R} \mapsto E_i (i = 1, 2)$ be functions which satisfy the relation

$$\xi_i(t) = V^E_i(t-\sigma) \xi_i(\sigma) + \lim_{n \to \infty} \int_\sigma^t V^E_i(t-s) \Pi^E_i (I^n f(s)) \, ds.$$

Then the $\mathcal{B}$-valued function $\xi$ defined as $\xi(t) = \xi_1(t) + \xi_2(t)$, $t \in \mathbb{R}$, satisfies the relation

$$\xi(t) = V(t-\sigma) \xi(\sigma) + \lim_{n \to \infty} \int_\sigma^t V(t-s) I^n f(s) \, ds \quad (\forall t \geq \sigma).$$

Hence Theorem 3.1 yields that

$$\xi(t) = u(t, \sigma; \xi(\sigma); f) \quad (\forall t \geq \sigma).$$

Now, in the remainder of the paper we always assume that $\mathcal{B}$ satisfies Axiom (A2) in addition to (A1). Then, by employing the same argument as in the proof of [10, Theorem 4.2.9], we obtain the following result:
Theorem 4.2. Assume that $\mathcal{B}$ is decomposed as above. If the functions $\xi_i : \mathbb{R} \to E_i$ $(i = 1, 2)$ satisfy the relation

$$
\xi_i(t) = V^{E_i}(t - \sigma) \xi_i(\sigma) + \lim_{n \to \infty} \int_0^t V^{E_i}(t - s) \Pi^{E_i}(\Gamma^n f(s)) \, ds,
$$

then the function $u(t)$ defined by $u(t) = [\xi_1(t) + \xi_2(t)](0)$ for $t \in \mathbb{R}$ is a solution of (2) on $\mathbb{R}$ and satisfies $u_i = \xi_i(t)$ in $\mathcal{B}$.

Next, we consider the case where the space $\mathcal{B}$ is decomposed as the direct sum of closed subspaces $S$ and $U$, where $S$ is the stable subspace for $V(t)$; that is

$$
\mathcal{B} = S \oplus U, \quad V(t) S \subset S, \quad V(t) U \subset U \quad (\forall t \geq 0)
$$

and

$$
\exists C > 0, \alpha > 0 : \|V^S(t)\| \leq Ce^{-\alpha t} \quad (\forall t \geq 0).
$$

In what follows, we shall establish the existence and uniqueness of a $\mathcal{B}$-valued function $y$ satisfying the equation associated with the $S$-component of the variation-of-constants formula in the phase space. To this end, we set

$$
y(t) = \lim_{n \to \infty} \int_0^t V^S(t - s) \Pi^S(\Gamma^n f(s)) \, ds.
$$

Proposition 4.1. The above-defined $y(t)$ is well defined as a $\mathcal{B}$-valued function which is $\mathcal{B}$-bounded on $\mathbb{R}$ and satisfies the equation

$$
y(t) = V^S(t - \sigma) y(\sigma) + \lim_{n \to \infty} \int_0^t V^S(t - s) \Pi^S(\Gamma^n f(s)) \, ds \quad (\forall t \geq \sigma). \quad (5)
$$

Moreover, if $\overline{y} : \mathbb{R} \to S$ is $\mathcal{B}$-bounded on $\mathbb{R}$ and satisfies (5), then $\overline{y}(t) \equiv y(t)$ for all $t \in \mathbb{R}$.

Proof. We first observe that the limit

$$
\lim_{n \to \infty} \int_0^t V^S(t - s) \Pi^S(\Gamma^n f(s)) \, ds =: \int_0^t V^S(t - s) \Pi^S(\Gamma^n f(s)) \, ds
$$
exists in $\mathcal{B}$. Indeed, if $\sigma_1 < \sigma_2 < t$, then
\[
\left| \int_{\sigma_1}^{t} V^s(t-s) \Pi^s(\Gamma^n f(s)) \, ds - \int_{\sigma_1}^{t} V^s(t-s) \Pi^s(\Gamma^m f(s)) \, ds \right| \\
\leq \int_{\sigma_1}^{t} C e^{-(t-s)} \| \Pi^s \| K_0 \| f \| \\
\leq (CK_0/\alpha) e^{-(t-\sigma_1)} \| \Pi^s \| \| f \| \to 0
\]
as $\sigma_2 \to -\infty$, where $K_0 = K(1)$. Thus, the limit exists in $\mathcal{B}$.

Now, for any positive integers $n$ and $m$, we have
\[
\left| \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^n f(s)) \, ds - \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^m f(s)) \, ds \right| \\
\leq \left| \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^n f(s)) \, ds \right| + \left| \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^m f(s)) \, ds \right| \\
\leq (2CK_0/\alpha) e^{(\sigma-\infty)} \| \Pi^s \| \| f \| \\
+ \left| \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^n f(s)) \, ds - \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^m f(s)) \, ds \right|
\]
Therefore it follows from Lemma 3.4 that
\[
\limsup_{n,m \to \infty} \left| \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^n f(s)) \, ds - \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^m f(s)) \, ds \right| \\
\leq (2CK_0/\alpha) e^{(\sigma-\infty)} \| \Pi^s \| \| f \| \to 0
\]
as $\sigma \to -\infty$, and hence the limit $\lim_{n,m \to \infty} \int_{-\infty}^{t} V^s(t-s) \Pi^s(\Gamma^n f(s)) \, ds$ exists in $\mathcal{B}$. If $t \geq \sigma$, then
\[
V^s(t-\sigma) y(\sigma) + \lim_{n \to \infty} \left| \int_{\sigma}^{t} V^s(t-s) \Pi^s(\Gamma^n f(s)) \, ds \right|
\leq \lim_{n \to \infty} V^s(t-\sigma) \left| \int_{-\infty}^{\sigma} V^s(\sigma-s) \Pi^s(\Gamma^n f(s)) \, ds \right|
+ \lim_{n \to \infty} \left| \int_{\sigma}^{t} V^s(t-s) \Pi^s(\Gamma^n f(s)) \, ds \right|
\]
\[
\lim_{n \to \infty} \left\{ \int_{-\infty}^{\tau} V^S(t-s) \Pi^S(\Gamma^nf(s)) \, ds + \int_{\tau}^{\infty} V^S(t-s) \Pi^S(\Gamma^nf(s)) \, ds \right\} = \lim_{n \to \infty} \int_{-\infty}^{\tau} V^S(t-s) \Pi^S(\Gamma^nf(s)) \, ds = y(t).
\]

Thus \(y(t)\) satisfies (5). Moreover, \(y\) is \(B\)-bounded because of

\[
|y(t)|_B \leq \sup_n \left| \int_{-\infty}^{\tau} V^S(t-s) \Pi^S(\Gamma^nf(s)) \, ds \right|_B \leq \int_{-\infty}^{\tau} Ce^{-\omega(t-s)} \, ds \|\Pi^S\| K_0 \|f\| = (K_0 C / \omega) \|f\|.
\]

If \(\bar{y}\) is another \(B\)-bounded function satisfying (5), then \(y(t) - \bar{y}(t) = V^S(t-s)(y(s) - \bar{y}(s))\) for all \(t \geq \sigma\). Hence

\[
|y(t) - \bar{y}(t)|_B \leq Ce^{-\omega(t-s)} \left\{ \sup_t |y(t)|_B + \sup_t |\bar{y}(t)|_B \right\} \to 0
\]
as \(\sigma \to -\infty\), and hence \(y(t) \equiv \bar{y}(t)\) in \(B\). This completes the proof.

Next we shall consider the case where the unstable subspace is finite dimensional. Let us assume that the space \(B\) is decomposed as

\[
B = S \oplus U, \quad V(t) S \subset S, \quad V(t) U \subset U,
\]

where \(S\) is the stable subspace for \(V(t)\) and \(U\) is finite dimensional. Let \(d = \dim U\). Take a basis \(\{\phi_1, ..., \phi_d\}\) in \(U\), and set \(\Phi = (\phi_1, ..., \phi_d)\). Then there exist \(d\)-elements \(\psi_1, ..., \psi_d\) in \(B^*\) (the dual space of \(B\)) such that \(\langle \psi_i, \phi_j \rangle = 1\) if \(i = j\) and \(0\) if \(i \neq j\) and that \(\psi_i = 0\) on \(S\). Here and hereafter, \(\langle , \rangle\) denotes the canonical pairing between the dual space and the original space. Denote by \(\Psi\) the transpose of \((\psi_1, ..., \psi_d)\) to use the expression \(\langle \Psi, \Phi \rangle = I_d\) (here \(I_d\) is the \(d \times d\) unit matrix). Then the projection operator \(\Pi^U\) is given by

\[
\Pi^U \phi = \Phi \langle \Psi, \phi \rangle, \quad \phi \in B.
\]

Since \((V^U(t))_{t \geq 0}\) is a strongly continuous semigroup on the finite dimensional space \(U\), it is norm continuous, so there exists a \(d \times d\) matrix \(G\) such that

\[
V^U(t) \Phi = \Phi e^{Gt} \quad (\forall t \geq 0).
\]
For any positive integer \( n \) and \( i \in \{1, \ldots, d\} \), we consider the functional \( \Gamma_n^i \) on \( X \) defined by

\[
\Gamma_n^i(x) = \langle \psi_i, \Gamma_n^i x \rangle, \quad x \in X.
\]

Then \( \Gamma_n^i \) is bounded linear on \( X \) with \( \| \Gamma_n^i \| \leq K(1) \| \psi_i \| \). Now let us define the \( d \)-column vector \( \Gamma_n^i \) in the dual space \( X^* \) of \( X \) as the transpose of \( (\Gamma_n^1, \ldots, \Gamma_n^d) \). Then \( \langle \Gamma_n^i, x \rangle = \langle \psi_i, \Gamma_n^i x \rangle \) and \( \sup_n \| \Gamma_n^i \| < K(1) \sup \| \psi_i \| < \infty \). Next, let \( z(t) \) be the component of \( \Pi^U u(t, \phi; f) \) with respect to the basis vector \( \Phi \); that is,

\[
\Pi^U u(t, \phi; f) = \Phi z(t).
\]

Then \( z(t) = \langle \psi, u(t, \phi; f) \rangle \) and the equation

\[
\Pi^U u(t, \phi; f) = \Pi^U (1) \Pi^U + \lim_{n \to \infty} \int_0^t \Pi^U (t-s) \Pi^U (\Gamma_n^i f(s)) ds \quad (t \geq \sigma)
\]

can be rewritten as

\[
z(t) = e^{G(t-\sigma)}z(\sigma) + \lim_{n \to \infty} \int_0^t e^{G(t-s)} \langle \Gamma_n^i, f(s) \rangle ds \quad (t \geq \sigma).
\] (6)

In fact, the following result will show that the equation associated with the unstable subspace \( U \) in the variation-of-constants formula is reduced to an ordinary differential equation.

**Proposition 4.2.** Let \( d = \dim U \). Then there exist a \( d \times d \) matrix \( G \) and a \( d \)-column vector \( \Gamma_n^i \) in \( X^* \) such that a \( U \)-valued function \( \zeta(t) \) is a solution of the equation

\[
\zeta(t) = \Pi^U (t-\sigma) \zeta(\sigma) + \lim_{n \to \infty} \int_0^t \Pi^U (t-s) \Pi^U (\Gamma_n^i f(s)) ds \quad (t \geq \sigma)
\]

if and only if the function \( z(t) \) determined by \( \Phi z(t) = \zeta(t) \) is a solution of the following ordinary differential equation

\[
\dot{z}(t) = Gz(t) + \langle \Gamma_n^i, f(t) \rangle
\] (7)

**Proof.** We first note that

\[
\Phi \cdot \lim_{n \to \infty} \int_0^t e^{G(t-s)} \langle \Gamma_n^i, f(s) \rangle ds = \Pi^U x(t, 0, f)
\] (8)
for any $X$-valued bounded continuous function $f$. Now, we assert that the sequence $\{x_n^*\}$ of $d$-column vectors in $X^*$ converges to a $d$-column vector $x^*$ in $X^*$ with respect to the weak-star topology; that is,

$$\lim_{n \to \infty} \langle x_n^*, x \rangle = \langle x^*, x \rangle, \quad \forall x \in X.$$ 

If this is the case, by applying Lebesgue’s convergence theorem we see that (6) can be rewritten as

$$z(t) = e^{G(t-s)}z(s) + \int_s^t e^{G(t-\sigma)}\langle x^*, f(\sigma) \rangle \, d\sigma \quad (t \geq \sigma),$$

and the conclusion follows from the argument in the paragraph preceding the proposition. Now, let $Y$ be any separable closed subspace of $X$. Since the sequence $\{x_n^*\}$ is bounded, $\{x_n^*\}$ contains a subsequence which converges with respect to the weak-star topology in $Y$, say, $\lim_{k \to \infty} \langle x_{n_k}^*, x \rangle = \langle x_Y^*, x \rangle$, $x \in Y$, for some column vector $x_Y^*$ in $Y^*$. We claim that

$$\lim_{n \to \infty} \langle x_n^*, x \rangle = \langle x_Y^*, x \rangle, \quad x \in Y. \tag{9}$$

Indeed, if this is not the case, there exist a subsequence $\{x_{n_k}^*\}$ and a column vector $x_Y^* \neq x_Y^*$, in $Y^*$ such that $\lim_{k \to \infty} \langle x_{n_k}^*, x \rangle = \langle x_Y^*, x \rangle$, $x \in Y$. For any $x \in Y$, set $f(\cdot) \equiv x$. Then, by (8) we get

$$\Phi \int_s^t e^{G(t-\sigma)}\langle x_Y^*, x \rangle \, d\sigma = \Phi \int_s^t e^{G(t-\sigma)}\langle x_Y^*, x \rangle \, d\sigma.$$

Hence, $\langle x_Y^*, x \rangle = \langle x_Y^*, x \rangle$ for all $x \in Y$, which is a contradiction to $x_Y^* \neq x_Y^*$. Thus, the claim must be true.

It follows from (9) that $\langle x_Y^*, x \rangle = \langle x_Y^*, x \rangle (x \in Y \cap Z)$ for any separable closed subspaces $Y$ and $Z$ of $X$. Now for any $x \in X$, we set

$$\langle x^*, x \rangle = \langle x_Y^*, x \rangle,$$

where $Y$ is any separable closed subspace of $X$ which contains $x$. Then $x^*$ is well defined on $X$. Moreover, we can see that $x^*$ is bounded linear on $X$ with $\|x^*\| \leq \sup_n \|x_n^*\| < \infty$ and $\lim_{n \to \infty} \langle x_n^*, x \rangle = \langle x^*, x \rangle$ for all $x \in X$, as required. This completes the proof. \[\square\]

Combining Propositions 4.1 and 4.2 with Theorem 4.2, we get the following result:
Theorem 4.3. Assume that the space $\mathcal{B}$ is decomposed as
\[ \mathcal{B} = S \oplus U, \quad V(t) S \subset S, \quad V(t) U \subset U, \]
where $S$ is the stable subspace for $V(t)$ and $U$ is finite dimensional. Let $G$, $\Phi$, $\Psi$, and $x^*$ be defined as for Proposition 4.2. Then, if $\xi(t)$ is a solution of (2) on $\mathbb{R}$, $z(t) := \langle \Psi, \xi \rangle$ is a solution of the ordinary differential equation (7) on $\mathbb{R}$. Conversely, if $z(t)$ is a solution of (7) on $\mathbb{R}$, then the function
\[ u(t) := \Phi z(t) + \lim_{\lambda \to \infty} \int_0^t V^S(t-\tau) H^S(t^*f(\tau)) \, d\tau \]
is a solution of (2) on $\mathbb{R}$.

5. APPLICATIONS TO THE STUDY OF THE ADMISSION OF FUNCTION SPACES

In this section, as an application of the results obtained above we shall study the admissibility of function spaces with respect to linear functional differential equations of the form
\[ \dot{u}(t) = Au(t) + L(u_t), \tag{10} \]
where $A$ is the infinitesimal generator of a strongly continuous compact semigroup on $X$, and $L$ is a bounded linear operator mapping a uniform fading memory space $\mathcal{B} = \mathcal{B}(\mathbb{R}^+; X)$ into $X$.

A closed subspace $\mathcal{M}$ of $BC(\mathbb{R}; X)$ is said to be admissible with respect to Eq. (10), if for any $f \in \mathcal{M}$, Eq. (2) possesses a unique solution which belongs to $\mathcal{M}$. We refer the reader to [4, 8, 11, 12, 14, 17–19, 22, 23, 27] and the references therein for more information of the theory of admissibility of function spaces with respect to linear equations in the bounded as well as the unbounded case. In what follows, we shall investigate the admissibility with respect to (10) for a translation-invariant space of bounded uniformly continuous functions whose spectra are contained in a closed subset of $\mathbb{R}$. In this direction our study here is closely related to the recent works [4, 17–19, 22, 23]. The main idea of our study is to use the decomposition of variation-of-constants formula to reduce the admissibility problem with respect to Eq. (10) to the one with respect to an ordinary differential equation. As a result, we get a necessary and sufficient condition for the admissibility which seems to be the sharpest possible for the class of functional differential equations of the form (10).
First, we recall the notion of a spectrum of a given function \( f \in BC(\mathbb{R}; X) \) which is defined as the set

\[
sp(f) := \{ \lambda \in \mathbb{R} : \forall \varepsilon > 0 \exists \chi \in L^1(\mathbb{R}), supp \ \tilde{\chi} \subset (\lambda - \varepsilon, \lambda + \varepsilon), \chi \ast f \neq 0 \},
\]

where \( L^1(\mathbb{R}) \) is the space of all complex-valued integrable functions on \( \mathbb{R} \), and

\[
(\chi \ast f)(t) := \int_{-\infty}^{+\infty} \chi(t-s) f(s) \, ds; \quad \tilde{\chi}(s) := \int_{-\infty}^{+\infty} e^{-ist} \chi(t) \, dt.
\]

We collect some main properties of the spectrum of a function for the reader’s convenience. For the proof we refer the reader to [13, 23, 26].

**Proposition 5.1.** The following statements hold true:

(i) \( sp(e^{i\lambda}) = \{ \lambda \} \) for \( \lambda \in \mathbb{R} \),

(ii) \( sp(e^{i\lambda} f) = sp(f) + \lambda \) for \( \lambda \in \mathbb{R} \),

(iii) \( sp(\alpha f + \beta g) \subseteq sp(f) \cup sp(g) \) for \( \alpha, \beta \in \mathbb{C} \),

(iv) \( sp(f) \) is closed; moreover, \( sp(f) \) is not empty if \( f \neq 0 \),

(v) \( sp(f(\cdot + \tau)) = sp(f) \) for \( \tau \in \mathbb{R} \),

(vi) If \( f, g^i \in BC(\mathbb{R}; X) \) with \( sp(g^i) \subset A \) for all \( n \in \mathbb{N} \), and if

\[
\lim_{k \to \infty} \|g^k - f\| = 0,
\]

then \( sp(f) \subset \overline{A} \).

(vii) \( sp(\chi \ast f) \subset sp(f) \cap supp \ \tilde{\chi} \) for all \( \chi \in L^1(\mathbb{R}) \).

For any closed set \( A \), we set

\[
A(X) = \{ f \in BC(\mathbb{R}; X) : sp(f) \subset A \}.
\]

In virtue of Proposition 5.1, we can see that \( A(X) \) is a translation-invariant closed subspace of \( BC(\mathbb{R}; X) \). In what follows, we will give a condition under which the subspace \( A(X) \) or \( A(X) \cap AP(X) \) is admissible with respect to Eq. (10), where \( AP(X) \) denotes the space of all \( X \)-valued almost periodic functions in the sense of Bohr.

For any \( \lambda \in \mathbb{R} \), we define a function \( \omega(\lambda) : \mathbb{R}^- \to \mathbb{C} \) by

\[
[\omega(\lambda)](\theta) = e^{i\lambda \theta}, \quad \theta \in \mathbb{R}^-.
\]
Because $\mathcal{B}$ is a uniform fading memory space, it follows that $\omega(\lambda) x \in \mathcal{B}$ for any $x \in X$. We denote by $I$ the identity operator on $X$ and define a linear operator $L(\omega(\lambda) I)$ on $X$ by

$$L(\omega(\lambda) I) x = L(\omega(\lambda) x), \quad x \in X.$$  

We can see that the operator $L(\omega(\lambda) I)$ is bounded.

Now, let $G$ be the infinitesimal generator of the solution semigroup $(V(t))_{t \geq 0}$, and consider the set $\Sigma_U = \{ \lambda \in \sigma(\mathcal{B}) : \Re \lambda \geq 0 \}$. Then $\Sigma_U$ is a finite set, and each point in $\Sigma_U$ does not belong to the essential spectrum of $\mathcal{B}$ (cf. [20, Theorem 11]). Corresponding to the set $\Sigma_U$, we get the decomposition

$$\mathcal{B} = S \oplus U,$$

where $S$ is the stable subspace for $V(t)$, and the unstable subspace $U$ for $V(t)$ is finite dimensional. Therefore, one can apply results given in the preceding section. In fact, the $\mathcal{B}$-valued function $y$ introduced in Section 4 satisfies the relation

$$(\chi * y)(t) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \chi(t-s) \left( \int_{0}^{\infty} V^S(\tau) \Pi^S(\tau f(s-\tau)) \, d\tau \right) ds$$

for any $\chi \in L^1([0, \infty))$. Therefore, we can see that the function $\eta$ defined by $\eta(t) = [y(t)](0)$ satisfies the relation $sp(\eta) \subseteq sp(f)$. In particular, from this fact and Theorem 4.3 we see that the admissibility of $A(X)$ and $A(X) \cap AP(X)$ with respect to (10) follows from the admissibility of the ordinary differential equation $\dot{z} = Gz(t)$, where the spectrum of the matrix $G$ is identical to the set $\Sigma_U$. Moreover, by virtue of [20, Theorem 12], we see that $i \lambda$ belongs to the set $\Sigma_U$ if and only if the characteristic operator $i \lambda I - L(\omega(\lambda) I) - A$ has a nontrivial null space. In fact, for any $s$ in the resolvent of $A$ we have the relation $[i \lambda I - L(\omega(\lambda) I) - A] (sI - A)^{-1} = I + K$ with $K = [(i \lambda - s) I - L(\omega(\lambda) I)] (sI - A)^{-1}$ compact, and hence the invertibility of $i \lambda I - L(\omega(\lambda) I) - A$ in $B(X)$ follows from the injectiveness of $i \lambda I - L(\omega(\lambda) I) - A$. Summarizing these facts, we arrive at the following result.

**Theorem 5.1.** Suppose $A$ generates a compact semigroup on $X$ and $\mathcal{B} = \mathcal{A}(\mathbb{R}^+; X)$ is a uniform fading memory space, and let $A$ be a closed set in $\mathbb{R}$. Then the following statements are equivalent:
(i) The space $A(X)$ is admissible respect to (10).

(ii) The space $A(X) \cap AP(X)$ is admissible with respect to (10).

(iii) $[(i\lambda I - L(\omega(\lambda) I) - A)^{-1} \in B(X)$ for any $\lambda \in A$.

Under one of the conditions (i)–(iii), there exists a function $F \in L^1(\mathbb{R}; B(X))$ such that

$$\tilde{F}(\lambda) = (i\lambda I - L(\omega(\lambda) I) - A)^{-1}$$

$$(\lambda \in A).$$

Moreover, $F * f$ is a unique solution of (2) in $A(X)$ for any $f \in A(X)$.

**Proof.** Using the above decomposition we can use the results and methods in [17, Theorem 3.11 and Corollary 3.5] or [18, Theorem 3.11 and Corollaries 2 and 3] to find the component of solution on the unstable subspace by solving an ordinary differential equation of the form (7). The other component is uniquely determined since on the stable subspace the solution semigroup is exponentially stable.

**ACKNOWLEDGMENTS**

The authors thank the referee for the remarks and for pointing out several references with related results. This work was supported in part by the Grants-in-Aid for Scientific Research (C), 12640155 (the first author Y. Hino), 11640191 (the second author S. Murakami), and 11640155 (the third author T. Naito) of the Japanese Ministry of Education, Science, Sport, and Culture.

**REFERENCES**


