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Stability and stabilization of switched linear dynamic systems with time delay and uncertainties

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ABSTRACT

This paper considers the problem of exponential stability and stabilization of switched linear time-delay systems. The system parameter uncertainties are time-varying and unknown but norm-bounded. The delay in the system states is also time-varying. By using an improved Lyapunov–Krasovskii functional, a switching rule for the exponential stability and stabilization is designed in terms of the solution of Riccati-type equations. The approach allows for computation of the bounds that characterize the exponential stability rate of the solution. Numerical examples are given to illustrate the results.

1. Introduction

Switching systems belong to an important class of hybrid systems, which are described by a family of differential equations together with specified rules to switch between them. A switching system can be represented by a differential equation of the form

$$\dot{x}(t) = f_\sigma(t,x), \quad t \geqslant 0,$$

where $\{f_\sigma(\cdot,\cdot) : \sigma \in \mathscr{I}\}$ is a family of functions parameterized by some index set $\mathscr{I}$, which is typically a finite set, and $\sigma(\cdot)$, which depends on the system state at each time, is the switching rule/signal determining a switching sequence for the given system.

Switching systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models, such as manufacturing, communication networks, automotive engineering control, chemical processes (e.g. see [5,7,14] and the references therein). In the last two decades, there has been increasing interest in stability analysis and control design for switched systems (e.g. [5,8,13,14,20]). Also, during the last decades, the stability problem of uncertain linear time-delay systems and applications to control theory has attracted a lot of attention [2–4,11,12]. The main approach for stability analysis relies on the use of Lyapunov–Krasovskii functionals and linear matrix inequality (LMI) for constructing suitable Lyapunov–Krasovskii functionals.

Although some important results have been obtained for linear switched systems, there are few results concerning the stability of switched linear systems with time delay and uncertainties. In [15], the problem of stabilization via state feedback and/or state-based switching for switched linear systems with multiple time-varying delays without uncertainties was considered. It was proved in [15] that the switched linear delay system will be stabilizable via state feedback and/or switching if the corresponding system with zero delays has a Hurwitz stable convex combination and the delays less than an appropriate

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upper bound that satisfies a set of LMIs. In [16,18], delay-dependent asymptotic stability conditions are extended to discrete-time linear switching systems with time delay. Considering switching systems composed of a finite number of linear point time-delay differential equations, it has been shown recently in [6], that the asymptotic stability may be achieved by using a common Lyapunov function method switching rule. There are some other results concerning asymptotic stability for switching linear systems with time delay, but most of them provide conditions for the asymptotic stability or stabilization of switched systems without focusing on exponential stability. The exponential stability problem was considered in [21] for switching linear systems with impulsive effects by using the matrix measure concept, and in [19] for nonholonomic chained systems with strongly nonlinear input/state driven disturbances and drifts. On the other hand, it is worth noting that the existing stability conditions for time-delay systems must be solved upon a grid of the parameter space, which results in testing a nonlinear Riccati-type equation or a finite number of LMIs. In this case, the results using finite gridding points are unreliable and the numerical complexity of the tests grows rapidly. Therefore, finding new conditions for the robust exponential stability of uncertain linear switching time-delay systems is of interest.

In this paper, we study the problem of robust exponential stability for a class of uncertain linear hybrid time-delay systems. Different from [6,15,19,21], the system considered in this paper is subject to time-varying uncertainties and time-varying delay. Our objective is to derive delay-dependent conditions for the exponential stability by using an improved Lyapunov–Krasovskii functional. The conditions will be presented in terms of the solution of Riccati-type equations. Comparing with the previous results, a simple geometric design is employed to find the switching rule and our approach allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. The result is applied to obtain new sufficient conditions for stabilization of linear uncertain control switching systems. The paper can be considered as an extension of existing results for linear switching time-delay systems.

The paper is organized as follows. Section 2 presents notations, definitions and a technical lemma required for the proof of the main results. Sufficient conditions for the exponential stability and application to stabilization together with illustrative examples are presented in Section 3. The paper ends with a conclusion followed by cited references.

2. Preliminaries

The following notations will be used throughout this paper. $\mathbb{R}^+$ denotes the set of all real non-negative numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $\langle\cdot,\cdot\rangle$ and the vector norm $\|\cdot\|$; $\mathbb{R}^{n\times r}$ denotes the space of all matrices of $(n\times r)$-dimensions. A^T denotes the transpose of A; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A)=\max\{\mathrm{Re}\,\lambda:\lambda\in\lambda(A)\}$; $\lambda_{\min}(A)=\min\{\mathrm{Re}\,\lambda:\lambda\in\lambda(A)\}$; A matrix A is semi-positive definite ($A\geqslant 0$) if $\langle Ax,x\rangle\geqslant 0$, for all $x\in\mathbb{R}^n$; A is positive definite ($A>0$) if $\langle Ax,x\rangle>0$ for all $x\neq 0$; $A\geqslant B$ means $A-B\geqslant 0$.

Consider a class of uncertain linear hybrid time-delay systems of the form

$$\begin{cases}\dot{x}(t)=[A_\sigma+\Delta A_\sigma(t)]x(t)+[D_\sigma+\Delta D_\sigma(t)]x(t-h(t)), & t\in\mathbb{R}^+,\\ (t)=\phi(t), & t\in[-h,0],\end{cases}\tag{2.1}$$

where $x(t)\in\mathbb{R}^n$ is the system state; $\sigma(\cdot):\mathbb{R}^n\to\mathscr{I}:=\{1,2,\ldots,N\}$ is the switching function, which is piece-wise constant function depending on the state at each time and will be designed. $A_\sigma,D_\sigma\in\{[A_i,D_i],i=1,2,\ldots,N\},A_i,D_i$ are given matrices and $\phi(t)\in C([-h,0],\mathbb{R}^n)$ is the initial function with the norm $\|\phi\|=\sup_{s\in[-h,0]}\|\phi(s)\|$. The uncertainties satisfy the following conditions:

$$\Delta A_i(t)=E_{0i}F_{0i}(t)H_{0i},\quad \Delta D_i(t)=E_{1i}F_{1i}(t)H_{1i},$$

where $E_{ki},H_{ki},k=0,1,i=1,2,\ldots,N$ are given constant matrices with appropriate dimensions; $F_{ki}(t)$ are unknown, real matrices satisfying

$$F_{ki}^T(t)F_{ki}(t)\leqslant I,\quad k=0,1,\ \ i=1,\ldots,N\ \ \forall t\geqslant 0.$$

The time-varying delay function $h(t)$ is assumed to satisfy the following condition:

$$0\leqslant h(t)\leqslant h,\quad \dot{h}(t)\leqslant\mu<1,\qquad t\geqslant 0,$$

where h and μ are given constants. This assumption means that the time delay may change from time to time but the rate of changing is bounded, i.e. the delay cannot increase as fast as the time itself.

Definition 2.1. Given $\beta>0$. The system (2.1) is β-exponentially stable if there exists a switching function $\sigma(\cdot)$ and positive number γ such that any solution $x(t,\phi)$ of the system satisfies

$$\|x(t,\phi)\|\leqslant\gamma e^{-\beta t}\|\phi\|\quad\forall t\in\mathbb{R}^+\tag{2.2}$$

for all the uncertainties.

Definition 2.2 [17]. The system of matrices $\{L_i\},i=1,2,\ldots,N$, is said to be strictly complete if for every $x\in\mathbb{R}^n\setminus\{0\}$ there is $i\in\{1,2,\ldots,N\}$ such that $x^TL_ix<0$.

Let us define

$$\Omega_i = \{x \in \mathbb{R}^n : x^T L_i x < 0\}, \quad i = 1, 2, \ldots, N.$$

It is easy to show that the system $\{L_i\}, i = 1, 2, \ldots, N$, is strictly complete if and only if

$$\bigcup_{i=1}^{N} \Omega_i = \mathbb{R}^n \setminus \{0\}. \tag{2.3}$$

Remark 2.1. As shown in [17], a sufficient condition for the strict completeness of the system $\{L_i\}$ is that there exist $\xi_i \geqslant 0, i = 1, 2, \ldots, N$ such that $\sum_{i=1}^{N} \xi_i > 0$ and

$$\sum_{i=1}^{N} \xi_i L_i < 0.$$

If $N = 2$ then the above condition is also necessary for the strict completeness.

Next, we introduce the following lemma, which will be used in the proof of our results.

Lemma 2.1 [11]. *For any $x, y \in \mathbb{R}^n$, matrices P, E, F, H with $P > 0, F^T F \leqslant I$, and scalar $\varepsilon > 0$, one has*

(1) $EFH + H^T F^T E^T \leqslant \varepsilon^{-1} EE^T + \varepsilon H^T H$,
(2) $2x^T y \leqslant x^T P^{-1} x + y^T P y$.

3. Main results

In the sequel, for the sake of brevity, we will denote σ for the switching signal $\sigma(\cdot)$.
For given numbers β, h, μ and symmetric positive definite matrix P we set

$$\tau = (1-\mu)^{-1}, \quad \eta = \tau e^{2\beta h} + 2\beta;$$

$$S_i = E_{0i} E_{0i}^T + e^{2\beta h} E_{1i} E_{1i}^T, \quad Q = \sum_{i=1}^{N} D_i^T P D_i, \quad R = \sum_{i=1}^{N} H_{1i}^T H_{1i},$$

$$L_i(P) = A_i^T P + P A_i + H_{0i}^T H_{0i} + P S_i P + Q + \tau R + \eta P, \tag{3.1}$$

$$\alpha_1 = \lambda_{\min}(P), \quad \alpha_2 = \lambda_{\max}(P) + h\left[\sum_{i=1}^{N} \lambda_{\max}(D_i^T P D_i) + \tau \sum_{i=1}^{N} \lambda_{\max}(H_{1i}^T H_{1i})\right]. \tag{3.2}$$

Theorem 3.1. *The system (2.1) is β-exponentially stable if there exists a symmetric positive definite matrix P such that the system of matrices $\{L_i(P)\}, i = 1, 2, \ldots, N$ is strictly complete.*

Moreover, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t,\phi)\| \leqslant \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\beta t} \|\phi\|, \quad t \in \mathbb{R}^+.$$

Proof. Consider the following Lyapunov–Krasovskii functional:

$$V(x_t) = V_1(x(t)) + V_2(x_t) + V_3(x_t),$$

where $x_t \in C([-h, 0], \mathbb{R}^n), x_t(s) = x(t+s), s \in [-h, 0]$ and

$$V_1(x(t)) = x^T(t) P x(t),$$

$$V_2(x_t) = \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) Q x(s) ds,$$

$$V_3(x_t) = \frac{1}{1-\mu} \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) R x(s) ds.$$

It is easy to verify that

$$\alpha_1 \|x(t)\|^2 \leqslant V(x_t) \leqslant \alpha_2 \|x_t\|^2, \quad t \geqslant 0, \tag{3.3}$$

where α_1, α_2 are respectively defined by (3.2).
Taking derivative of $V_1(x(t)) = x^T(t) P x(t)$ along trajectories of any subsystem ith we have

$$\dot{V}_1(x(t)) = x^T(t)[A_i^T P + P A_i] x(t) + 2x^T(t) P \Delta A_i(t) x(t) + 2x^T(t) P D_i x(t-h(t)) + 2x^T(t) P \Delta D_i(t-h(t)).$$

Applying Lemma 2.1 gives

$$\begin{aligned} &2x^T(t)P\Delta A_i(t)x(t) \leqslant x^T(t)PE_{0i}E_{0i}^TPx(t) + x^T(t)H_{0i}^TH_{0i}x(t), \\ &2x^T(t)P\Delta D_ix(t-h(t)) \leqslant e^{2\beta h}x^T(t)PE_{1i}E_{1i}^TPx(t) + e^{-2\beta h}x^T(t-h(t))H_{1i}^TH_{1i}x(t-h(t)), \\ &2x^T(t)PD_ix(t-h(t)) \leqslant \tau e^{2\beta h}x^T(t)Px(t) + \tau^{-1}e^{-2\beta h}x^T(t-h(t))D_i^TPD_ix(t-h(t)). \end{aligned} \tag{3.4}$$

Next, taking derivative of $V_2(x_t)$ and $V_3(x_t)$, respectively, along the system trajectories yields

$$\begin{aligned} \dot V_2(x_t) &= -2\beta V_2(x_t) + x^T(t)Qx(t) - (1-\dot h(t))x^T(t-h(t))e^{-2\beta h(t)}Qx(t-h(t)) \\ &\leqslant -2\beta V_2(x_t) + x^T(t)Qx(t) - \tau^{-1}e^{-2\beta h}x^T(t-h(t))Qx(t-h(t)), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \dot V_3(x_t) &= -2\beta V_3(x_t) + \tau x^T(t)Rx(t) - \tau(1-\dot h(t))x^T(t-h(t))e^{-2\beta h(t)}Rx(t-h(t)) \\ &\leqslant -2\beta V_3(x_t) + \tau x^T(t)Rx(t) - e^{-2\beta h}x^T(t-h(t))Rx(t-h(t)). \end{aligned} \tag{3.6}$$

From (3.1), (3.4)–(3.6) we get

$$\dot V(x_t) + 2\beta V(x_t) \leqslant x^T(t)[A_i^TP + PA_i + Q + \tau R + PS_iP]x(t) + \eta x^T(t)Px(t) + x^T(t)H_{0i}^TH_{0i}x(t) = x^T(t)L_i(P)x(t). \tag{3.7}$$

Let us set

$$\Omega_i(P) = \{x \in \mathbb{R}^n : x^TL_i(P)x < 0\}.$$

Then by the strict completeness of the system of matrices $\{L_i(P)\}$, and from (2.3) it follows that

$$\bigcup_{i=1}^N \Omega_i(P) = \mathbb{R}^n \setminus \{0\}.$$

Defining the sets

$$\widetilde\Omega_1(P) = \Omega_1(P), \quad \widetilde\Omega_i(P) = \Omega_i(P) \setminus \bigcup_{j=1}^{i-1}\widetilde\Omega_j(P), \ \ i = 2,3,\ldots,N,$$

we see that

$$\bigcup_{i=1}^N \widetilde\Omega_i(P) = \mathbb{R}^n \setminus \{0\}, \quad \widetilde\Omega_i(P) \cap \widetilde\Omega_j(P) = \emptyset, i \neq j.$$

Therefore, for any $x(t) \in \mathbb{R}^n, t \geqslant 0$, there exists $i \in \{1,2,\ldots,N\}$ such that $x(t) \in \widetilde\Omega_i(P)$. By choosing switching rule as $\sigma(x(t)) = i$ whenever $x(t) \in \widetilde\Omega_i(P)$, from (3.7) we have

$$\dot V(x_t) + 2\beta V(x_t) \leqslant x^T(t)L_i(P)x(t) \leqslant 0, \quad t \geqslant 0.$$

This implies that $V(x_t) \leqslant V(\phi)e^{-2\beta t}, t \geqslant 0$. Taking (3.3) into account, we obtain

$$\alpha_1\|x(t,\phi)\|^2 \leqslant V(x_t) \leqslant V(\phi)e^{-2\beta t} \leqslant \alpha_2 e^{-2\beta t}\|\phi\|^2, \qquad t \geqslant 0$$

and then

$$\|x(t,\phi)\| \leqslant \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\beta t}\|\phi\|, \quad t \geqslant 0,$$

which concludes the proof of the Theorem 3.1. □

Remark 3.1. Note that by Remark 2.1, the system $\{L_i(P)\}$ is strictly complete if there exist $\xi_i \geqslant 0, i = 1,2,\ldots,N, \sum_{i=1}^N \xi_i > 0$ such that

$$\sum_{i=1}^N \xi_iL_i(P) < 0. \tag{3.8}$$

In this case, the switching rule can be chosen as

$$\sigma(x(t)) = \arg\min\{x^T(t)L_i(P)x(t)\}, \quad t \geqslant 0.$$

Indeed, as shown in the proof of Theorem 3.1, we have arrived at the estimation

$$\dot V(x_t) + 2\beta V(x_t) \leqslant x^T(t)L_i(P)x(t), \quad t \geqslant 0.$$

Since $\xi_i \geqslant 0$ and $\xi = \sum_{i=1}^N \xi_i > 0$, so

$$\min_{i=1,2,\ldots,N} x^T(t)L_i(P)x(t) \leqslant \xi^{-1}\sum_{i=1}^N \xi_ix^T(t)L_i(P)x(t).$$

By choosing switching rule as

$$\sigma(x(t)) = \arg\min\{x^T(t)L_i(P)x(t)\}, \quad t \geqslant 0,$$

we have

$$\dot{V}(x_t) + 2\beta V(x_t) \leqslant x^T(t)L_i(P)x(t) \leqslant \xi^{-1}\sum_{i=1}^{N}\xi_i x^T(t)L_i(P)x(t) \leqslant 0.$$

This leads to

$$\|x(t,\phi)\| \leqslant \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\beta t}\|\phi\|, \quad t \geqslant 0$$

as desired.

The following procedure can be applied to design the switching rule:

Step 1: Define the symmetric positive definite matrix P (i.e. the solution of the matrix inequality (3.8)) such that the system $\{L_i(P)\}$ is strictly complete.
Step 2: Construct the sets $\Omega_i(P)$, and then $\widetilde{\Omega}_i(P)$.
Step 3: The switching rule is chosen as $\sigma(x(t)) = i$, whenever $x(t) \in \widetilde{\Omega}_i(P)$.

Example 1. Consider the system (2.1), where $N = 2, h(t) = 0.5\sin^2 t$ and

$$[A_1, D_1] = \left[\begin{pmatrix} -20 & 1 \\ -4 & 6 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right],$$
$$[A_2, D_2] = \left[\begin{pmatrix} 5 & -1 \\ 1 & -30 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}\right],$$
$$E_{0i} = E_{1i} = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad H_{0i} = H_{1i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that, both matrices A_1 and A_2 are unstable. In this case, we have $h = 0.5$, $\mu = 0.5, \tau = 2, \beta = 1$. We verify that the symmetric positive definite matrix

$$P = \begin{pmatrix} 3.3922 & -1.5840 \\ -1.5840 & 1.8170 \end{pmatrix}$$

satisfies (3.8) with $\xi_1 = \xi_2 = 0.5$, that is,

$$L(P) = 0.5L_1(P) + 0.5L_2(P) < -0.5I,$$

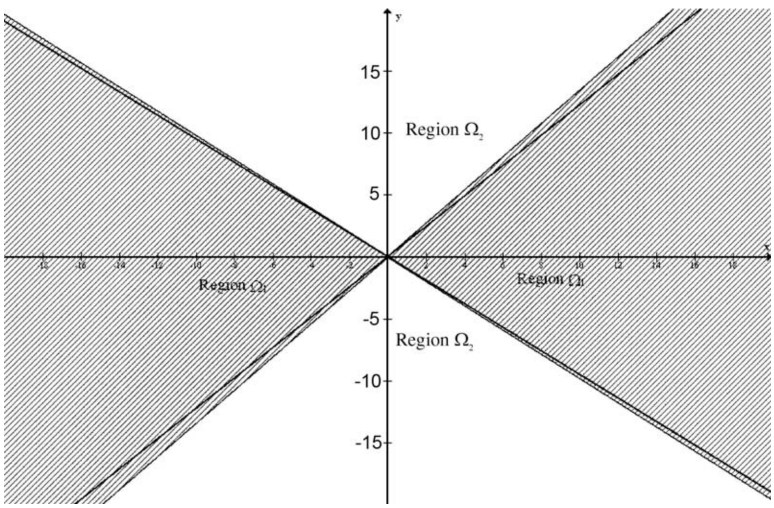

Fig. 1. Regions of Ω_1, Ω_2.

where

$$L_1(P) = \begin{pmatrix} -78.4218 & -10.8561 \\ -10.8561 & 59.8458 \end{pmatrix}, \quad L_2(P) = \begin{pmatrix} 75.3468 & 8.8683 \\ 8.8683 & -64.6418 \end{pmatrix}.$$

Therefore, the system $\{L_1(P), L_2(P)\}$ is strictly complete. The sets $\Omega_1(P), \Omega_2(P)$ are defined as

$$\Omega_1(P) = \{(x,y) \in \mathbb{R}^2 : -78.4218x^2 - 21.7122xy + 59.8458y^2 < 0\},$$
$$\Omega_2(P) = \{(x,y) \in \mathbb{R}^2 : 75.3468x^2 + 17.7366xy - 64.6418y^2 < 0\},$$

which can be represented in Fig. 1.

It can be seen that $\Omega_1(P) \cup \Omega_2(P) = \mathbb{R}^2 \setminus \{0\}$. Therefore, the switching regions are given as

$$\widetilde{\Omega}_1(P) = \{(x,y) \in \mathbb{R}^2 : -78.4218x^2 - 21.7122xy + 59.8458y^2 < 0\};$$
$$\widetilde{\Omega}_2(P) = \{(x,y) \in \mathbb{R}^2 : -78.4218x^2 - 21.7122xy + 59.8458y^2 \geqslant 0, (x,y) \neq (0,0)\}.$$

We have $\widetilde{\Omega}_1(P) \cup \widetilde{\Omega}_2(P) = \mathbb{R}^2 \setminus \{0\}, \widetilde{\Omega}_1(P) \cap \widetilde{\Omega}_2(P) = \emptyset$. The switching rule is chosen as

$$\sigma(x(t)) = \begin{cases} 1 & \text{if } x(t) \in \widetilde{\Omega}_1(P), \\ 2 & \text{if } x(t) \in \widetilde{\Omega}_2(P). \end{cases}$$

By Theorem 3.1, the solution of the system satisfies

$$\|x(t,\phi)\| \leqslant 5.2885e^{-t}\|\phi\| \quad \forall t \geqslant 0.$$

For the case when $N = 1$ (without switching), Theorem 3.1 gives an exponential estimate for the robust stability of uncertain linear time-delay systems, as considered in [9,10].

Corollary 3.1. *The uncertain linear time-delay system*

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [D + \Delta D(t)]x(t - h(t)), \quad t \geqslant 0 \tag{3.9}$$

is β-exponentially stable if there exists a symmetric positive definite matrix P such that the following condition hold:

$$A^T P + PA + D^T PD + PSP + \eta P + M < 0,$$

where $S = E_0E_0^T + e^{2\beta h}E_1E_1^T, M = H_0^TH_0 + \tau H_1^TH_1, \eta = 2\beta + \tau e^{2\beta h}$.

Moreover, the solution of the system (3.9) satisfies

$$\|x(t,\phi)\| \leqslant \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\beta t}\|\phi\|, \quad t \in \mathbb{R}^+,$$

where $\alpha_1 = \lambda_{\min}(P), \alpha_2 = \lambda_{\max}(P) + h[\lambda_{\max}(D^TPD) + \tau\lambda_{\max}(H_1^TH_1)]$.

For comparison with the condition obtained in [9,10], we consider the following example.

Example 2. Consider the linear uncertain time-delay system

$$\dot{x}(t) = [A + \Delta A]x(t) + [D + \Delta D]x(t - h),$$

where

$$A = \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}, \quad D = \begin{pmatrix} 0.1 & 0 \\ 4 & 0.1 \end{pmatrix}, \quad \|\Delta A\| \leqslant 0.2, \ \|\Delta D\| \leqslant 0.2.$$

Here $h = 0.5, \mu = 0, E_0 = E_1 = 0.2I, H_0 = H_1 = I$, then Corollary 3.1 gives the decay rate $\beta = 0.9539$ and the stability factor $\gamma = 5.9053$ with the solution matrix P

$$P = 1.0e + 005 * \begin{pmatrix} 8.4328 & 2.7162 \\ 2.7162 & 1.6256 \end{pmatrix}$$

and the solution satisfies

$$\|x(t,\phi)\| \leqslant 5.9053e^{-0.9539t}, \quad t \geqslant 0.$$

It is interesting to note that the decay rate for this system by using Corollary 3.1 is greater than decay rate $\beta = 0.476$, obtained by using Theorem 2 in [9] or $\beta = 0.095$ from the matrix measure results in [10].

As an application of Theorem 3.1, we consider stabilization problem of a linear switching control time-delay system of the form

$$\begin{cases} \dot{x}(t) = [A_\sigma + \Delta A_\sigma(t)]x(t) + [D_\sigma + \Delta D_\sigma(t)]x(t - h(t)) + [B_\sigma + \Delta B_\sigma(t)]u(t), & t \in \mathbb{R}^+, \\ x(t) = \phi(t), & t \in [-h, 0], \end{cases} \tag{3.10}$$

where $u(t) \in \mathbb{R}^m$ is the control; $B_\sigma \in \{[B_i], i = 1, 2, \ldots, N\}, B_i$ are given constant matrices. The uncertainty $\Delta B_i(t)$ satisfies:

$$\Delta B_i(t) = E_{2i} F_{2i}(t) H_{2i}, \quad i = 1, 2, \ldots, N, \ \ t \in R^+,$$

where $E_{2i}, H_{2i}, i = 1, 2, \ldots, N$ are given constant matrices with appropriate dimensions.

Definition 3.1. Given $\beta > 0$. The system (3.10) is β-exponentially stabilizable if there exist matrices $K_i \in R^{m \times n}$ such that the resulting closed-loop system

$$\dot{x}(t) = [A_\sigma + B_\sigma K_\sigma + \Delta A_\sigma(t) + \Delta B_\sigma(t) K_\sigma] x(t) + [D_\sigma + \Delta D_\sigma(t)] x(t - h(t)) \tag{3.11}$$

is β-exponentially stable. The control $u(t) = K_\sigma x(t)$ is stabilizing feedback control of the system.

To proceed with the exponential stabilization condition, we set

$$\widetilde{S}_i = E_{0i} E_{0i}^T + E_{2i} E_{2i}^T + e^{2\beta h} E_{1i} E_{1i}^T,$$
$$\widetilde{L}_i(P) = A_i^T P + P A_i - P B_i B_i^T P + H_{0i}^T H_{0i} + \frac{1}{4} P B_i H_{2i}^T H_{2i} B_i^T P + P \widetilde{S}_i P + Q + \tau R + \eta P,$$

where

$$Q = \sum_{i=1}^{N} D_i^T P D_i, \quad R = \sum_{i=1}^{N} H_{1i}^T H_{1i}.$$

Theorem 3.2. *The system (3.10) is β-exponentially stabilizable if there exists a symmetric positive definite matrix P such that one of the following conditions holds:*

(i) *The system matrices $\{\widetilde{L}_i(P)\}$ is strictly complete.*
(ii) *There exist $\xi_i \geqslant 0, \sum_{i=1}^{N} \xi_i > 0$ such that*

$$\sum_{i=1}^{N} \xi_i \widetilde{L}_i(P) < 0. \tag{3.12}$$

The switching rule is defined as $\sigma(x(t)) = i$ whenever $x(t) \in \widetilde{\Omega}_i$ in case (i), *and as*

$$\sigma(x(t)) = \arg\min\{x^T(t) \widetilde{L}_i(P) x(t)\}, \quad t \geqslant 0,$$

in case (ii). *The feedback stabilizing control is given by $u(t) = -\frac{1}{2} B_i^T P x(t), t \geqslant 0$.*

Proof. For the feedback control $u(t) = K_i x(t)$, where $K_i = -\frac{1}{2} B_i^T P$, we define

$$\widetilde{A}_i = A_i + B_i K_i, \quad \widetilde{E}_{0i} = (\,E_{0i} \quad E_{2i}\,),$$
$$\widetilde{F}_{0i}(t) = \begin{pmatrix} F_{0i}(t) & 0 \\ 0 & F_{2i}(t) \end{pmatrix}, \quad \widetilde{H}_{0i} = \begin{pmatrix} H_{0i} \\ H_{2i} K \end{pmatrix}.$$

Note that

$$E_{0i} F_{0i}(t) H_{0i} + E_{2i} F_{2i}(t) H_{2i} = (\,E_{0i} \quad E_{2i}\,) \begin{pmatrix} F_{0i}(t) & 0 \\ 0 & F_{2i}(t) \end{pmatrix} \begin{pmatrix} H_{0i} \\ H_{2i} K \end{pmatrix},$$

the closed-loop system (3.11) becomes

$$\dot{x}(t) = [\widetilde{A}_i + \widetilde{E}_{0i} \widetilde{F}_{0i}(t) \widetilde{H}_{0i}] x(t) + [D_i + \Delta D_i(t)] x(t - h(t)), \quad t \geqslant 0.$$

Therefore, the proof of Theorem 3.2 is then completed by the same arguments used in the proof of Theorem 3.1. □

Remark 3.2. It was proved in [15] that the switched linear delay system without uncertainties will be stabilizable via state feedback and/or switching if the corresponding system with zero delays has a Hurwitz stable convex combination and the delays less than an appropriate upper bound that satisfies a set of LMIs. Theorem 3.2 provide sufficient conditions for robust exponential stability and stabilization of uncertain linear switching systems with time-varying delay.

Remark 3.3. The delay-dependent conditions for the exponential stability and stabilization are derived in terms of the solution of Riccati-type inequalities (3.8) and (3.12). To find the solution of these Riccati inequalities, one can use various computationally efficient techniques, for example, in [1].

Example 3. Consider the switched uncertain time-delay control system (3.10), where $h(t) = 1.5\sin^2(0.6t)$, and

$$[A_1, D_1, B_1] = \left[\begin{pmatrix} -20 & 1 \\ -4 & 6 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right],$$

$$[A_2, D_2, B_2] = \left[\begin{pmatrix} 5 & -1 \\ 1 & -30 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}\right],$$

$$E_{0i} = E_{1i} = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$H_{0i} = H_{1i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_{21} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad H_{22} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Here, we have $h = 1.5, \mu = 0.9$ and $\tau = 10, \beta = 0.5$. The condition (3.12) gives

$$\widetilde{L}(P) = 0.5\widetilde{L}_1(P) + 0.5\widetilde{L}_2(P) < 0,$$

where

$$P = \begin{pmatrix} 547.6711 & -49.7510 \\ -49.7510 & 24.8041 \end{pmatrix}.$$

The feedback control can thus be obtained as $u(t) = K_i x(t)$, where

$$K_1 = -\frac{1}{2}B_1^T P = [\,-522.7956 \quad 37.3489\,],$$

$$K_2 = -\frac{1}{2}B_2^T P = [\,-697.1292 \quad 12.6161\,].$$

By using Theorem 3.2, the uncertain switching control system (3.10) is exponentially stabilizable and the solution of the system satisfies

$$\|x(t,\phi)\| \leqslant 9.979e^{-0.5t}\|\phi\| \quad \forall t \geqslant 0.$$

4. Conclusion

This paper has proposed a switching design for the exponential stability and stabilization of uncertain linear switching time-delay systems. The stability conditions are derived in terms of the solution of Riccati-type equations. The approach allows for the use of efficient techniques for computation of the two bounds that characterize the exponential stability rate of the solution, as well as the feedback control.

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